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Critical renormalized coupling constants in the symmetric phase of the Ising models

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Abstract. Using a finite-size scaling method combined with extensive Monte Carlo measurements of high statistics, we calculate the four-, six- and eight-point renormalized coupling constants defined at zero momentum in the symmetric phase of the three-dimensional Ising system. The results of the 2D Ising system that are directly measured are also reported. Our value of the four-point coupling constant for the 3D system agrees very well with the available estimates from other methods. Our values of the six- and eight-point coupling constants are significantly different from those obtained by other methods for the 3D system, although they agree reasonably well in the 2D system.

1. Introduction

The Hamiltonian of the Ising ferromagnet is given by

$$H = -\sum_{\langle i,j\rangle} S_i S_j \tag{1}$$

where the Ising spin at site i, S_i, can take either 1 or -1 and the sum is over all the nearest neighbours of the lattice. It is well known that the critical behaviour of the D-dimensional Ising model can be described by the D-dimensional Euclidean scalar field theory in which the Hamiltonian is given by

$$H = \int \mathrm{d}^D x \left[\frac{1}{2}(\nabla\phi(x))^2 + \frac{1}{2}m_0^2\phi(x)^2 + \frac{g_0}{4!}\phi(x)^4\right] \tag{2}$$

where m_0 and g_0 are respectively the bare mass and the coupling constant defined in the absence of critical fluctuations of the fields. Near the criticality m_0 is a linear measure of temperature, and we denote by $m_{0\mathrm{c}}$ the value at the critical point. As the fluctuations become strong, renormalizations of the mass, coupling constant and fields are necessary, and the long-distance behaviour is no longer described by the bare potential but by the effective potential which is generally of more complicated functional form than the bare one. In statistical physics the effective potential represents the free-energy density as a function of order parameter (expectation value of the renormalized field) and is used to determine the equation of state.

After small-renormalized-field expansion of the effective potential, its coefficients are directly related to the renormalized coupling constants (RCCs) defined at zero momentum. In terms of the expectation value of the renormalized field, φ_R, the effective potential in the 3D symmetric phase may be written as

$$V_{\mathrm{eff}}(\varphi_R) = \frac{1}{2}m_R^2\varphi_R^2 + \frac{1}{4!}m_R g_R^{(4)}\varphi_R^4 + \frac{1}{6!}g_R^{(6)}\varphi_R^6 + \frac{1}{8!}\frac{g_R^{(8)}}{m_R}\varphi_R^8 + \cdots \tag{3}$$

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where m_R and $g_R^{(2N)}$ represent respectively the renormalized mass (inverse correlation length ξ) and the (dimensionless) $2N$-point RCC defined at zero momentum.

The formal expression of $g_R^{(2N)}$ can be obtained by calculating the $2N$th derivative of the effective potential with respect to the average value of the renormalized field. Necessary elements for the calculations are the well known relations

$$\frac{\mathrm{d}V_{\mathrm{eff}}}{\mathrm{d}\varphi} = J \qquad \varphi = \frac{1}{V}\frac{\mathrm{d}W[J]}{\mathrm{d}J} \qquad \text{and} \qquad \varphi = Z_\varphi^{1/2}\varphi_R. \tag{4}$$

Here φ, V, W, and Z_φ are respectively the expectation value of the bare field ϕ, the volume of the system, the generating functional for connected Green functions in the presence of the external field J, and the field strength renormalization factor given by $Z_\varphi = \chi m_R^2$ with χ denoting magnetic susceptibility. The expressions in 3D are

$$g_R^{(4)} = -(Z_\varphi^2/m_R)W_2^{-4}W_4 \tag{5}$$

$$g_R^{(6)} = -Z_\varphi^3 W_2^{-6}[W_6 - 10W_4^2 W_2^{-1}] \tag{6}$$

$$g_R^{(8)} = -m_R Z_\varphi^4 W_2^{-8} \times [W_8 - 56W_6W_4W_2^{-1} + 280W_4^3W_2^{-2}] \tag{7}$$

where W_N is the Fourier transformed N-point connected Green function at zero momentum. In 2D, since φ_R is dimensionless, the expressions for the $g_R^{(2N)}$ values are given by equations (5)–(7) with additional division of the m_R^2 term in the right-hand sides of them.

In statistical physics, the effective potential is used to determine the equation of state, that is, the induced magnetization as a function of external field through the first of equations (4). In this sense the role of higher-order RCCs becomes important for increasing values of the magnetization. Each RCC has a universal non-zero finite value as the theory becomes critical, provided the hyperscaling relation holds. An RCC can also be expressed as a combination of dimensionless amplitude ratios which is again universal, e.g., $g_R^{(6)} = (g_R^{(4)})^2(10 - R_0^{-1})$ with R_0 denoting the universal sixth-order ratio [1]. An accurate determination of a universal quantity is generally important in statistical physics because it characterizes a universality class. In other words, the value of an RCC is supposed to be the same for any class of theories having the same symmetry and dimension. In computing an RCC we take thermodynamic limit first and then take the limit of the thermodynamic correlation length becoming divergent (that is, the limit $m_0 \to m_{0c}$). The corresponding value in this limit may be termed as critical RCC. The determination of the accurate value of the critical four-point RCC ($\tilde{g}_R^{(4)}$) is particularly important since all other universal quantities can be given in terms of it.

In this work we are concerned with the Monte Carlo calculation of the critical RCCs in the symmetric phase of the two- and three-dimensional Ising model. (From here on the notation of RCC will be used without the subscript R when it is used in context of size dependence.) This has been a subject of many studies, including a variety of quantum field theoretic approaches [2–7], high-temperature series expansions [8–10], exact renormalization group flow techniques [11–13], and Monte Carlo simulations [14–16]. It is known from the previous Monte Carlo simulation based on the Metropolis algorithm that standard direct Monte Carlo measurements of the thermodynamic critical RCCs of higher order ($N \geqslant 3$) suffer from an enormous statistical noise [14]. We are not aware of any previous Monte Carlo studies on the higher-order RCCs of the 2D Ising model in the symmetric phase although there are some very recent results available from field-theoretic approaches [6, 7]. Another Monte Carlo study for the 3D Ising model is based on finding the probability distribution of the order parameter in the external field at various temperatures [15]. We employ a finite-size scaling (FSS) extrapolation scheme [17, 18] combined with a single-cluster-flipping Monte Carlo algorithm [19].

Preliminary Monte Carlo results for the 3D Ising model were reported in [20], which showed significant differences in the estimates of $g_R^{(6)}$ and $g_R^{(8)}$ from those obtained by other methods. The reason for the discrepancy must be clarified since any employed methods do not appear to be completely free of unambiguity in computing higher-order RCCs. This motivated us to carry out very extensive and thorough Monte Carlo simulations. We also tried to find a possible source of errors in the Monte Carlo simulation: (i) by increasing the precision of the Monte Carlo simulation, (ii) by extending our Monte Carlo measurements of RCCs to the 2D Ising model, and (iii) by including in our analysis the effect of the correction to scaling.

(i) is helpful for a precise check of the validity of the finite size scaling for the RCCs and (ii) helps in detecting a possible systematic error in our measurements of RCC by comparing those with available exact results at criticality. One way to determine the effect of correction to scaling is to check the correction of FSS by measurements of the RCCs at criticality, which are included in this study for both the 2D and 3D Ising models.

We observe that our Monte Carlo data are highly self-consistent among themselves and that an involvement of any systematic errors in our Monte Carlo measurements is unlikely to happen. This study essentially confirms the results in [20] for the 3D Ising model, while showing substantial agreement with those from other methods for the 2D Ising model.

2. The Monte Carlo method

For the calculations of the RCC one needs to calculate ξ, χ, and W_{2N}. With periodic boundary conditions imposed on the lattice, χ and W_{2N} can be expressed in terms of the expectation values of various powers of the sum of the spin over all the lattice site $S \equiv \sum_i S_i$. For example,

$$\chi = \langle S^2 \rangle / L^D \qquad W_4 = (\langle S^4 \rangle - 3\langle S^2 \rangle^2)/L^D \tag{8}$$

where L is the linear size of the lattice. The ξ can be very accurately determined using the standard second moment formula†. The cluster algorithms [19] of the Monte Carlo simulation have been extremely efficient for many problems of critical phenomena. It is now a relatively easy task to obtain Monte Carlo data of typical physical quantities such as χ and ξ with relative statistical errors less than 0.5% at a temperature arbitrarily close to criticality. It nevertheless turns out that an accurate Monte Carlo measurement of a higher-order RCC is problematic in some cases. The problem arises basically from the fact that a RCC of higher order is given as a multiplication of a huge number with a tiny one. The former is given typically by some power of $\frac{L}{\xi_L}$, for instance, $(\frac{L}{\xi_L})^9$ for the $g_R^{(8)}$ of the 3D system, whereas the latter comes from the combination of the W_{2N} that turns out to be extremely sensitive to statistical noises. The noises increase rapidly with increasing temperature and increasing value of $\frac{L}{\xi_L}$.

As an illustration we measured various physical quantities at an arbitrary (inverse) temperature in the scaling regime of the 2D Ising model with increasing linear size of the lattice L. Table 1 clearly shows that the statistical error in the measured values of $g_R^{(2N)}$ increases with N for any given value of L. In the case of $L = 80$, for example, the relative statistical errors of ξ_L, χ_L, $g_L^{(4)}$, $g_L^{(6)}$, $g_L^{(8)}$ are 0.08, 0.1, 0.7, 3.1, 5.3%, respectively. We observe that all the variables possibly except for $g_L^{(8)}$ are monotonically increasing functions of L up to $L = 80$ where $\frac{L}{\xi_L} \simeq 6.7$. The values do not vary with further increasing L within the statistical errors. We recall that the well known exact thermodynamic value of ξ at this β is $11.9055\ldots$, which manifests itself at $L = 80$ already within the statistical errors. The size-independent value thus corresponds to the thermodynamic value. It is also very clear that the size-dependence becomes rapidly weaker with increasing L for all the variables considered here. This is expected to be the case for any physical variable that has a well-defined thermodynamic value. For the $g_L^{(8)}$, however, owing to the large statistical noises it is hard to draw a definite conclusion on the

† See, for example, [18].

Table 1. Size dependence of various physical quantities at $\beta = 0.420$ up to $L = 100$ for the 2D Ising system.

L	ξ_L	χ_L	$g_L^{(4)}$	$g_L^{(6)}$	$g_L^{(8)}$
20	9.664(8)	116.1(1)	6.29(1)	218.46(1.32)	17 871(140)
30	10.977(9)	162.5(1)	8.95(1)	414.8 (2.3)	43 532(313)
40	11.53(1)	186.5(2)	11.26(1)	615.1(4.0)	71 443(615)
50	11.78(1)	197.4(2)	12.83(3)	740.2(5.5)	85 462(879)
60	11.84(1)	201.3(2)	13.79(4)	794.8(6.2)	87 919(1185)
70	11.88(1)	203.1(2)	14.20(6)	825.1(10.0)	89 824(2315)
80	11.91(1)	203.9(2)	14.77(10)	850.4(26.1)	89 521(4765)
90	11.91(2)	204.3(2)	14.85(12)	858.8(24.2)	90 722(7451)
100	11.90(2)	204.4(3)	14.60(16)	846(38)	88 165(9575)

size dependence. Nevertheless, it is very natural to expect that $g_L^{(8)}$ would follow a similar size dependence as $g_L^{(4)}$ and $g_L^{(6)}$.

To compute the critical RCC it is required to obtain thermodynamic values as $\beta \to \beta_c$. For the case of the 2D Ising model we will not follow that procedure. Instead, we will report the results of measurements at the critical point and try to find out the possible effect of the correction to scaling. The illustration in the case of the 2D Ising model turns out to be useful in connection with the study of the 3D case. Note that the relation in the scaling regime

$$g_R^{(2N)}(t) \sim t^{D\nu - 2\Delta_{2N} + \gamma} \tag{9}$$

translates into

$$g_L^{(2N)}(t=0) \sim L^{(D\nu - 2\Delta_{2N} + \gamma)/\nu} \tag{10}$$

at criticality. Hence the hyperscaling relation $D\nu - 2\Delta_{2N} + \gamma = 0$ implies the invariance of $g_L^{(2N)}(t=0)$ with respect to L. Conversely, from the (weak) dependence of $g_L^{(2N)}(t=0)$ one can sometimes infer the correction to scaling in the scaling regime. Computing the RCC by taking the limit $t \to 0$ first and then taking the limit $L \to \infty$ is equivalent to the computation of $g_L^{(2N)}(t=0)$. It should be stressed that $g_L^{(2N)}(t=0)$ basically represents the opposite of thermodynamic limit (that is, measurements under the condition $L/\xi_L \simeq 1$) and has nothing to do with the lower-bound of critical RCC [21].

We also report the sixth- and eighth-order cumulant ratios at criticality that are denoted by $U_L^{(6)}$ and $U_L^{(8)}$, respectively. These are part of W_6 and W_8 and have already been available for the 2D Ising model based on the umbrella sampling method of the Monte Carlo simulation [22]. Since our code is generic in dimension, the comparison with the result from a different Monte Carlo method for these higher-order cumulant ratios that have severe statistical noises may be useful to check against any unexpected possible systematic errors in our Monte Carlo measurements of RCCs. For example, $U_L^{(8)}$ is expressed as

$$U_L^{(8)} = (\langle S^8\rangle - 28\langle S^6\rangle\langle S^2\rangle - 35\langle S^4\rangle^2 + 420\langle S^4\rangle\langle S^2\rangle^2 - 630\langle S^2\rangle^4)/\langle S^2\rangle^4. \tag{11}$$

The results are summerized in table 2. Our results of $U_L^{(2N)}$ for $N = 2, 3$, and 4 are respectively 1.8318(5), 13.93(1), and −226.0(1), which may be compared with 1.834, 13.96, and −226.6 reported in [22]. The exact values obtained very recently by Salas and Sokal are 1.832 0771(47), 13.936 806(71), and −226.0796(16), respectively†. We observe that the values of $g_L^{(2N)}(t=0)$ has no L dependence at least for $L \geqslant 40$, confirming the hyperscaling

† What the authors of [23] actually calculated is $V_{2N} \equiv \langle S^{2N}\rangle/\langle S^2\rangle^N(t=0)$ from which the cumulant ratios defined in the present study can be obtained.

Table 2. Size dependence of various physical quantities at criticality $\beta_c = \ln(\sqrt{2}+1)/2$ up to $L = 100$ for the 2D Ising system.

L	$U_L^{(4)}$	$U_L^{(6)}$	$U_L^{(8)}$	$g_L^{(4)}$	$g_L^{(6)}$	$g_L^{(8)}$
20	1.8324(6)	13.94(1)	−226.1(2)	2.227(7)	29.03(18)	933(9)
40	1.8321(6)	13.94(1)	−226.1(2)	2.237(8)	29.28(21)	945(10)
60	1.8317(5)	13.93(1)	−225.9(1)	2.241(7)	29.37(18)	949(9)
80	1.8318(5)	13.93(1)	−226.0(1)	2.240(6)	29.36(16)	948(8)
100	1.8316(6)	13.93(1)	−226.0(1)	2.239(7)	29.33(18)	947(9)

Table 3. Size dependence of the various physical quantities at $\beta = 0.217$ (the upper part) and $\beta = 0.220$ (the lower part). Note that the $g_L^{(8)}$ data for $\frac{L}{\xi_L} \geqslant 5.7$ become unreliable owing to huge error bars and the weaker size dependence for larger L. This appears to be the case even for $g_L^{(6)}$ for $L = 36$.

L	ξ_L	χ_L	$g_L^{(4)}$	$g_L^{(6)}$	$g_L^{(8)} \times 10^{-4}$
8	3.93(0)	59.49(6)	9.86(2)	492.5(1.9)	5.45(3)
12	4.85(1)	94.16(12)	13.59(3)	852(4)	10.8(1)
16	5.299(1)	114.54(5)	17.71(2)	1277(3)	15.77(9)
20	5.488(3)	124.20(7)	21.07(5)	1562(9)	15.5(3)
24	5.573(2)	128.40(5)	23.16(8)	1735(22)	15.5(9)
28	5.605(4)	130.06(11)	24.72(11)	1887(53)	14.6(3.8)
32	5.622(2)	130.87(5)	24.88(14)	2190(286)	13.3(16.9)
36	5.619(19)	130.96(14)	25.64(16)	—	—
16	7.85(2)	228.8(7)	9.52(5)	452.1(4.6)	—
20	8.85(2)	298.1(7)	11.43(5)	625.6(5.1)	—
24	9.56(2)	351.8(1.1)	13.44(7)	814.7(8.2)	—
30	10.20(3)	407.1(1.1)	16.5(1)	1123(16)	—
36	10.56(2)	439.2(1.2)	19.2(1)	1372(27)	—
40	10.68(3)	455.2(1.5)	21.4(2)	1501(56)	—
50	10.83(3)	467.9(1.3)	23.7(3)	1795(109)	—
60	10.89(3)	472.3(1.8)	24.5(4)	1983(148)	—
70	10.90(3)	473.0(1.1)	25.8(1.6)	—	—

and the absence of significant correction to scaling in the model. This is consistent with the previous Monte Carlo measurement of $g_R^{(4)}$ in the scaling regime which showed little variation in the value with respect to temperature change [16]. We were therefore led to the conclusion that the thermodynamic values of RCC measured at $\beta = 0.420$ may well be regarded as the critical values.

With respect to the size dependence we observe a similar feature in the 3D Ising model as in the 2D model. Our data are summerized in table 3. For the $\beta = 0.217$ and 0.220 we observe no size dependence (within the statistical errors) for the ξ and χ beyond $L = 32$ and 60, respectively. This roughly corresponds to $\frac{L}{\xi_L} \simeq 5.5$. We expect that this is the case even to the RCCs, although it was almost impossible to get precise measurements of the thermodynamic values of $g_L^{(6)}$ and $g_L^{(8)}$ for this value of $\frac{L}{\xi_L}$. For example, for $\beta = 0.217$ and $L = 32$ we generated about 10^9 single cluster sweeps, but error bars are larger than the mean value for the $g_L^{(8)}$. The values of $g_L^{(6)}$ for $L \geqslant 32$ also seem to be unreliable in view of the generic feature of the weaker size dependence with larger L.

It thus appears that Monte Carlo computation of the critical RCCs relying on direct brute-force measurements is prohibitively difficult. In order to overcome the difficulty in the close neighbourhood of T_c, we make use of a FSS function $\mathcal{Q}_A(x(L,t))$, defined by the

Table 4. Size dependence of the various physical quantities at $\beta = 0.221$ up to $\frac{L}{\xi_L} \simeq 4.11$.

L	ξ_L	χ_L	$g_L^{(4)}$	$g_L^{(6)}$	$g_L^{(8)} \times 10^{-4}$
20	10.95(2)	426.7(1.4)	7.6(1)	301(3)	2.67(4)
28	13.93(6)	703.4(4.0)	9.2(1)	423(8)	4.3(1)
36	15.98(6)	944.2(4.7)	11.2(1)	600(10)	6.6(2)
40	16.74(5)	1045.5(4.3)	12.4(1)	710(13)	8.2(2)
48	17.91(3)	1210(3)	14.59(6)	920(8)	10.8(2)
56	18.60(1)	1316(1)	16.74(5)	1116(7)	12.2(2)
64	19.03(2)	1386(2)	18.66(7)	1297(13)	13.4(4)
72	19.28(5)	1426(6)	20.47(29)	1465(57)	13.3(2.6)
80	19.46(2)	1458(2)	21.36(16)	1580(82)	—

expression [17, 18]

$$A_L(t) = A(t)\mathcal{Q}_A(x(L,t)) \qquad x(L,t) \equiv \frac{\xi_L(t)}{L}. \tag{12}$$

Here $A_L(t)$ represents the quantity A measured on a finite lattice of linear size L at a reduced temperature t, with its corresponding thermodynamic value $A(t)$. What equation (12) states is that the size dependence of a physical quantity A is given as a function of the scaling variable x. As a result, the ratio of L to ξ_L beyond which the thermodynamic limit is reached is independent of the temperature, which was shown to be approximately 5.5 for the 3D Ising model.

The FSS technique is especially useful for our purpose, because it enables us to extract accurate thermodynamic values based on the Monte Carlo measurements with much smaller lattices. We just outline the single-step FSS extrapolation technique used in this work. For a detailed explanation, we refer the reader to [17, 18].

(1) For a certain t_0, measure $A_L(t_0)$ and $x(L,t_0) = \xi_L(t_0)/L$ for increasing L.
(2) Determine the thermodynamic value at the temperature $A(t_0)$ by measuring $A_L(t_0)$ which is L independent.
(3) Fit $(x(L,t_0), A_L(t_0)/A(t_0))$ data to a suitable functional form. In this work we used the ansatz,

$$\mathcal{Q}(x) = 1 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 \tag{13}$$

(4) For any other t, choose a suitable L, measure the value of $x(L,t) \equiv \frac{\xi_L}{L}$ and $A_L(t)$, and interpolate $\mathcal{Q}(x(L,t))$.
(5) Extract $A(t)$ by inserting $A_L(t)$ and $\mathcal{Q}(x(L,t))$ into equation (12).

The smallest value of L we considered for the FSS method is 20. The boundary condition imposed for all our simulation is the periodic boundary condition. The numbers of single-clusters generated by the single-cluster algorithm for given values of L and β are typically of order 10^7 and 10^8, respectively, for the 2D and 3D Ising models.

3. Result and discussion

Our choice of β_0 is $\beta_0 = 0.220$ for the 3D Ising model. We infer from the generic feature of the size dependence observed for the 3D Ising model that thermodynamic limits of the RCCs are reached for $\frac{L}{\xi_L} \gtrsim 5.5$. We thus get $g_R^{(4)} = 24.5(4)$ and $g_R^{(6)} = 1983(148)$ at this β. Fitting

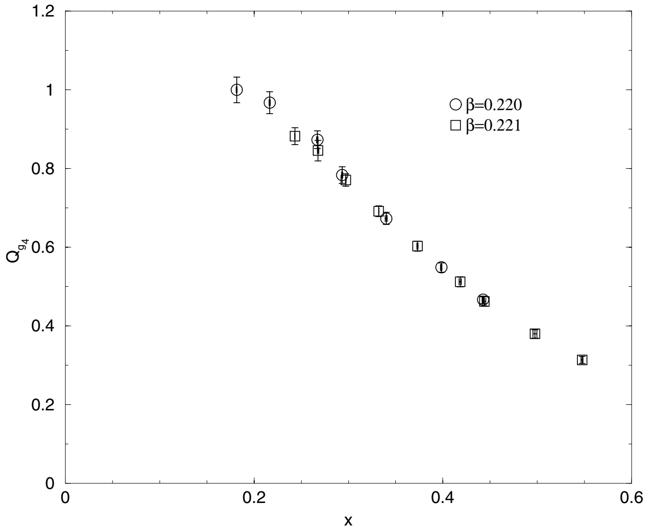

Figure 1. $\mathcal{Q}_{g^{(4)}}$ calculated for $\beta = 0.220$ and 0.221. The two data set collapse unto a single universal curve, showing numerical evidence for the FSS equation (12) for $g_R^{(4)}$.

the data to the ansatz (13), we get for $x \leqslant 0.4425(10)$

$$c_1 = 2.338 \qquad c_2 = -15.768 \qquad c_3 = 19.770 \qquad c_4 = -5.123$$
$$c_1 = 3.225 \qquad c_2 = -25.046 \qquad c_3 = 42.485 \qquad c_4 = -23.177$$

for the scaling function $\mathcal{Q}(x)$ of the $g^{(4)}$ and $g^{(6)}$, respectively. Using the scaling function we calculated the thermodynamic values of the 4- and 6-point RCCs for all the values of L from 36 to 80 in the table 4. The result from each choice of L is in reasonably good agreement: we get $g_R^{(4)} = 24.3(1)$, 24.4(4), 24.1(1), 23.9(1), 23.9(1), 24.1(3), and 23.9(2) for each L from the $L = 36$ through the $L = 80$ in the table, whereas for $g_R^{(6)}$ we get 1919(11), 1939(20), 1939(21), 1915(10), 1906(17), 1917(70), and 1897(49). The invariance of the thermodynamic RCC with respect to the choice of L is a numerical proof of the FSS for the variables (see figures 1 and 2). As usual, we extracted the thermodynamic value for several different choices of L for a given temperature and took the average. Our net results from $\beta = 0.217$ to $\beta = 0.2213$ are found in table 6. It is observed that both $g_R^{(4)}$ and $g_R^{(6)}$ tend to decrease mildly as $\beta \to \beta_c$. In this work we assume the widely accepted correction to scaling exponent [25] $\theta \simeq 0.5$ and $\beta_c = 0.221\,654$. By fitting our data in table 6 to

$$g_R^{(2N)}(t) = \tilde{g}_R^{(2N)}(1 + a_{2N}t^{0.5}) \tag{14}$$

we obtain the values of the critical RCCs which read

$$\tilde{g}_R^{(4)} = 23.6(2) \tag{15}$$

$$\tilde{g}_R^{(6)} = 1879(50). \tag{16}$$

Our results of $g_L^{(2N)}(t = 0)$ are presented in table 5. It is observed that both $U_L^{(4)}$ and $\frac{\xi_L}{L}$ have a very mild tendency of decreasing with increasing value of L. All the values of $g_L^{(2N)}(t = 0)$ ($N = 2, 3$, and 4) show remarkable invariance with respect to increasing L at least for $L \geqslant 30$. In other words, they do not show the effect of correction to scaling observed in the scaling regime, as is the case in the 2D Ising model. This is a slightly surprising

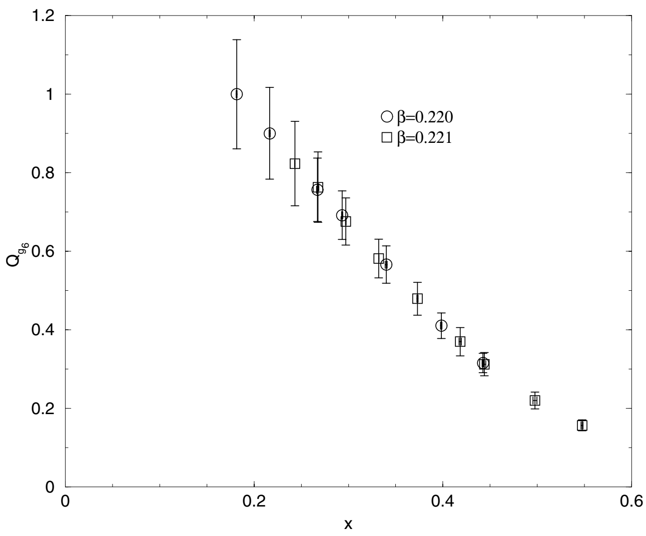

Figure 2. The same as in figure (1) but for $Q_{g^{(6)}}$.

Table 5. Size dependence of various physical quantities at criticality $\beta_c = 0.221\,654$ up to $L = 80$ for the 3D Ising system. Here $U_L^{(4)}$ represents the fourth-order cumulant ratio.

L	$U_L^{(4)}$	ξ_L	χ_L	$g_L^{(4)}$	$g_L^{(6)}$	$g_L^{(8)} \times 10^{-4}$
20	1.418(1)	12.80(1)	545.1(5)	5.40(1)	157.4(8)	1.07(1)
30	1.409(1)	19.20(2)	1215(1)	5.37(1)	155.3(7)	1.04(1)
40	1.408(2)	25.65(4)	2148(4)	5.34(2)	153.5(1.1)	1.03(1)
50	1.403(1)	31.98(4)	3313(4)	5.36(2)	154.4(1.0)	1.03(1)
60	1.401(2)	38.33(6)	4742(10)	5.37(2)	155.1(1.1)	1.04(1)
70	1.399(3)	44.69(8)	6418(19)	5.38(2)	155.2(1.2)	1.04(1)
80	1.398(2)	51.13(9)	8351(19)	5.36(2)	154.1(1.3)	1.03(1)

Table 6. Thermodynamic values of the four and six point RCC extracted by the FSS technique for some temperatures over $0.217 \leqslant \beta \leqslant 0.2213$. Here the quoted errors are obtained by ignoring the statistical errors in the estimate of the thermodynamic values at $\beta = 0.220$. Crudely speaking, increases in the estimate of $\beta = 0.220$ would lead to a more or less similar amount of increase in the estimate for other temperatures.

β	0.217	0.219	0.220	0.2206	0.2210	0.2213
$g_R^{(4)}$	25.2(2)	24.7(2)	24.5(0)	24.2(2)	24.2(2)	24.1(2)
$g_R^{(6)}$	2034(69)	2063(46)	1983(0)	1988(81)	1919(51)	1943(42)

result. There may be a few possible interpretations for the discrepancy. First let us recall that equation (14) translates into

$$g_L^{(2N)}(t=0) = \tilde{g}^{(2N)}(t=0)(1 + b_{2N} L^{-\omega}) \qquad (\omega > 0) \tag{17}$$

and that the coefficient b_{2N} may happen to be very small. The second possibility is that the currently accepted β_c might be slightly underestimated: at the exact criticality it is expected from the conformal field theory that the value of $\frac{\xi_L}{L}$ is a constant for modestly large values of L irrespective of the presence of the correction, which seems not to be the case at $\beta = 0.221\,654$.

Our current result at criticality may be summarized as follows:

$$\tilde{g}^{(4)}(t=0) = 5.36(2) \tag{18}$$

$$\tilde{g}^{(6)}(t=0) = 154.6(1.1) \tag{19}$$

$$\tilde{g}^{(8)}(t=0) = 1.035(10) \times 10^4. \tag{20}$$

Although the estimates, equations (18)–(20), would vary slightly depending on the choice of the value of the criticality, the near invariance of those values with respect to L almost certainly rule out any unexpected errors in our Monte Carlo measurements of the 3D Ising model as well.

It is impossible to apply the FSS method to the $g_R^{(8)}$ owing to its large error bars. Nevertheless, it is highly likely that at least $\tilde{g}_R^{(8)}$ is of order 10^5 for the following reasons. (i) From tables 3 and 4, it is evident that the values of $g_R^{(8)}$ are significantly larger than 10^5 up to $\frac{L}{\xi_L} \simeq 4$ where they can be precisely measured. (ii) We have accurate data at $\beta = 0.2213$ (not shown in this work), i.e., $g_L^{(8)} \simeq 1.13(3) \times 10^5$ already for $\frac{L}{\xi_L} \simeq 2.7$. (iii) It is observed that the ratio of critical RCCs to $g_L^{(2N)}(t=0)$ tends to increase with increasing N; the ratio is roughly 4.4 and 12.2 for $N = 2$ and 3, respectively, for the 3D and the value of $g_L^{(8)}(t=0)$ is already of order 10^4. (iv) We note that the L dependence of $g_L^{(8)}(t=0)$ is quantitatively the same as that of the other RCCs, indicating that the correction to scaling for $g_R^{(8)}(t)$ is as mild as that for the other RCCs. All the evidence almost certainly points out that the the value of the critical $g_R^{(8)}$ is much larger than those estimated by other methods. Our crude estimate of $\tilde{g}_R^{(8)}$ is

$$\tilde{g}_R^{(8)} \simeq 1.4(3) \times 10^5. \tag{21}$$

Our estimate of $\tilde{g}_R^{(4)}$, equation (15), is in reasonable agreement with other estimates. The agreement is especially good with the result from high-temperature expansion [8] and the field theoretic treatment [4,26]. However, the agreement becomes worse as N increases: the results of $\tilde{g}_R^{(6)}$ from previous studies are within the range $860 \lesssim \tilde{g}_R^{(6)} \lesssim 1515$. The closest result to ours, $\tilde{g}_R^{(6)} \simeq 1515$, obtained from a previous Monte Carlo method [15] is remarkable because it is not obtained by the direct measurement of RCC but by the Monte Carlo measurement of the probability distribution of the order parameter assuming the thermodynamic limit for $\frac{L}{\xi_L} \simeq 4$. We observe from tables 3 and 4 that the values of $\tilde{g}_R^{(6)}$ are approximately 1600 under the condition $\frac{L}{\xi_L} \simeq 4$. Since its critical value would, of course, be something slightly reduced, we conjecture that the result from the previous Monte Carlo method [15] would agree much better with our estimate if the same thermodynamic limit as in this work were taken. The values of $\tilde{g}_R^{(8)}$ from previous studies range from 2.9×10^4 to 3.5×10^4, which is at least four times smaller than our estimate. Although it was pointed out [8] that longer series terms are necessary for a more accurate estimate of higher-order critical RCC with the use of the high-temperature series expansion method, it remains puzzling why the previous studies based on different methods other than the Monte Carlo gave rise to results reasonably close to each other. Nevertheless, in view of limitations of almost all the methods used to study this subject and in light of the substantial consistency among Monte Carlo studies, it is fair to say that the issue is still open to further studies.

Our results of the critical RCCs for 2D Ising model obtained assuming a negligibly small correction to scaling read

$$\tilde{g}_R^{(4)} = 14.7(2) \tag{22}$$

$$\tilde{g}_R^{(6)} = 850(25) \tag{23}$$

$$\tilde{g}_R^{(8)} = 8.9(5) \times 10^4. \tag{24}$$

Our estimate of $\tilde{g}_R^{(4)}$ is in excellent agreement with the results reported in [7, 8, 10, 16, 26]. The estimate of $\tilde{g}_R^{(6)}$ and $\tilde{g}_R^{(8)}$ agree reasonably well with other estimates [6, 10, 24] that read

$$\tilde{g}_R^{(6)} \simeq 794 \qquad \tilde{g}_R^{(8)} \simeq 8.2(2) \times 10^4. \tag{25}$$

Our results at criticality are summerized as

$$\tilde{g}^{(4)}(t = 0) = 2.239(7) \tag{26}$$

$$\tilde{g}^{(6)}(t = 0) = 29.34(20) \tag{27}$$

$$\tilde{g}^{(8)}(t = 0) = 947(10). \tag{28}$$

These agree perfectly well within the statistical errors with the exact results calculated by Salas and Sokal [23], which are 2.236 6587(57), 29.254 57(15), and 942.6095(72), respectively.

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References

[1] Watson P G 1969 *J. Phys. C: Solid State Phys.* **2** 1883
[2] Bender C M *et al* 1980 *Phys. Rev. Lett.* **45** 501
Bender C M and Boettcher S 1993 *Phys. Rev.* D **48** 4919
[3] Sokolov A I 1996 *Fiz. Tverd. Tela* **38** 640
Sokolo A I, Ul'kov V A and Orlov E V 1997 *Phys. Lett.* A **227** 255
[4] Guida R and Zinn-Justin J 1997 *Nucl. Phys.* B **489** 626
[5] Campostrini M, Pelissetto A, Rossi P and Vicari E 1996 *Nucl. Phys.* B **459** 207
(Campostrini M, Pelissetto A, Rossi P and Vicari E 1999 *Preprint* cond-mat/9905078)
[6] Sokolov A I and Orlov E V 1998 *Phys. Rev.* B **58** 2395
[7] Jug G and Shalaev B N 1999 *J. Phys. A: Math. Gen.* **32** 7249
[8] Butera P and Comi M 1996 *Phys. Rev.* B **54** 15 828
Butera P and Comi M 1998 *Phys. Rev.* B **58** 11 552
[9] Reisz T 1995 *Nucl. Phys.* B **450** 569
[10] Zinn S Y, Lai S N and Fisher M E 1996 *Phys. Rev.* E **54** 1176
[11] Bagnuls C and Bervillier C 1990 *Phys. Rev.* B **41** 402
[12] Berges J, Tetradis N and Wetterich C 1996 *Phys. Rev. Lett.* **77** 873
[13] Morris T 1997 *Nucl. Phys.* B **495** 4777
[14] Wheater J F 1984 *Phys. Lett.* B **136** 402
[15] Tsypin M M 1994 *Phys. Rev. Lett.* **73** 2015
[16] Kim J-K and Patrascioiu A 1993 *Phys. Rev.* D **47** 2588
[17] Kim J-K 1994 *Phys. Rev.* D **50** 4663
[18] Kim J-K, de Souza A J F and Landau D P 1996 *Phys. Rev.* E **54** 2291
[19] Wolff U 1989 *Phys. Rev. Lett.* **62** 361
Swendsen R H and Wang J-S 1987 *Phys. Rev. Lett.* **58** 86
[20] Kim J-K and Landau D P 1997 *Nucl. Phys.* B (Proc. Suppl.) **53** 706
[21] Baker G A Jr and Kawashima N 199 *Phys. Rev. Lett.* **75** 994
Kim J-K 1996 *Phys. Rev. Lett.* **76** 2402
[22] Mon K K 1997 *Phys. Rev.* B **55** 38
[23] Salas J and Sokal A D 2000 *J. Stat. Phys.* **98** 551
(Salas J and Sokal A D 1999 *Preprint* cond-mat/9904038)
[24] Pelissetto A and Vicari E 1998 *Nucl. Phys.* B **522** [FS] 605
[25] See, for example, Guida R and Zinn-Justin J 1998 *J. Phys. A: Math. Gen.* **31** 8103 and references therein
[26] Pelissetto A and Vicari E 1998 *Nucl. Phys.* B **519** 626